School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 4th

1. Solve the differential equation

$$4\frac{d^2y}{dt^2}+\frac{dy}{dt}+16y=16\sin 2t \qquad y=9 \quad \frac{dy}{dt}=0 \quad t=0$$

Identify the transient part of the solution and the steady-state part. What is the amplitude of the steady-state solution? What is the phase of the steady-state solution?

We have to arrange the equation into the standard form

$$\frac{1}{2^2}\frac{d^2y}{dt^2}+\frac{2}{2\times 16}\frac{dy}{dt}+y=\sin 20t \qquad y=0.1 \quad \frac{dy}{dt}=0 \quad t=0$$

so

$$\omega_n=2, \qquad \zeta=1/16, \quad KF_0=1, \quad \omega=2$$

The steady state part of the solution follows as

$$x_p(t)=X_0\sin\left(\omega t+\phi\right)$$

$$X_0=\frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}=8 \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}=\frac{\pi}{2}$$

The transient part is

$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{x_0^h\cos\omega_d t+\frac{v_0^h+\varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

$$x_0^h=x_0-C-x_p(0)=x_0-C-X_0\sin\phi=9-8=1$$

$$v_0^h=v_0-\left.\frac{dx_p}{dt}\right|_{t=0}=v_0-X_0\omega\cos\phi=0$$

$$\omega_d=2\sqrt{1-\frac{1}{16^2}}=\sqrt{255}/8$$

Hence

$$x_h(t)=\exp(-t/8)\left\{\cos(\sqrt{255}t/8)+\frac{1}{\sqrt{255}}\sin(\sqrt{255}t/8)\right\}$$

The full solution is thus

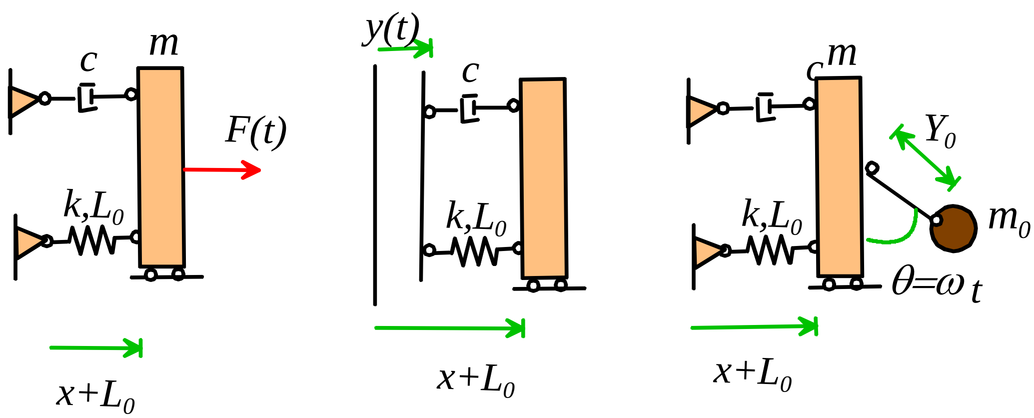
$$y=8\sin\left(\omega t+\frac{\pi}{2}\right)+\exp(-t/8)\left\{\cos(\sqrt{255}t/8)+\frac{1}{\sqrt{255}}\sin(\sqrt{255}t/8)\right\}$$

The first term is the steady-state solution; the second is transient (note that for large *t* the second term becomes zero)

[4 POINTS]

2. Determine the steady-state amplitude of vibration for the spring-mass systems shown in the figure (you don't need to derive the equations of motion – these are standard textbook systems and you can just use the standard formulas). In each case the mass *m=10*kg, the stiffness *k=1000N/m* *c=20Ns/m*.

The force is $F(t) = 20\sin 20t$; the base excitation is $y(t) = \sin 20t$, the length of the rotor is 0.25m; the eccentric mass $m_0 = 4kg$ and the angular velocity of the rotor is $\omega = 20$ rad/s.



The formulas for the amplitude yield, for the first system:

$$K = 1/k = 1/1000 \qquad F_0 = 20 \qquad \omega_n = \sqrt{k/m} = 10 \qquad \zeta = c/(2\sqrt{km}) = 0.1 \qquad \omega/\omega_n = 2$$

$$X_0 = \frac{KF_0}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} = 0.0066m$$

For the second system

$$K = 1 \quad Y_0 = 1 \quad \omega_n = \sqrt{k/m} = 10 \qquad \zeta = c/(2\sqrt{km}) = 0.1 \qquad \omega/\omega_n = 2$$

$$X_0 = \frac{KY_0\left\{1+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} = 0.356m$$

For the third system

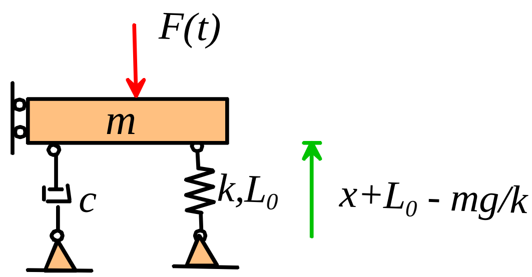
$$K = m_0/(m_0+m) = 4/14 \quad Y_0 = 1/4 \quad \omega_n = \sqrt{k/(m+m_0)} = 8.45$$

$$\zeta = c/(2\sqrt{k(m+m_0)}) = 0.0845 \qquad \omega/\omega_n = 2.367$$

$$X_0 = \frac{KY_0\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}} = 0.0866$$

[6 POINTS – 2 EACH]

3. The specifications for a force plate designed for jump performance analysis can be found at this website. A force-plate is essentially a spring-mass system – the force acting on the plate is determined by measuring the change in length of the spring x (the spring is actually a piezoelectric transducer). The force is then estimated by multiplying the length change by the spring stiffness kx . This works perfectly for a static force, but if the force applied to the plate varies with time, then $kx(t) \neq F(t)$. Suppose that:



(i) The natural frequency of the force-plate is 150 Hz

(ii) The system is critically damped (why is this a good choice for measuring a static force?)

(iii) The force applied to the plate is $F(t) = F_0 \sin\omega t$.

(iv) The amplitude of the force reading on the force-plate is $F_0^* = kX_0$ where X_0 is the vibration amplitude.

Critical damping is desirable for static readings because the force reading will then settle to its steady value after someone steps on the scale in the shortest possible time.

3.1 Write down the equation of motion for the x (substitute known numbers into the ‘standard form’ for a forced spring-mass system. Leave the spring stiffness as an unknown.)

The equation of motion is

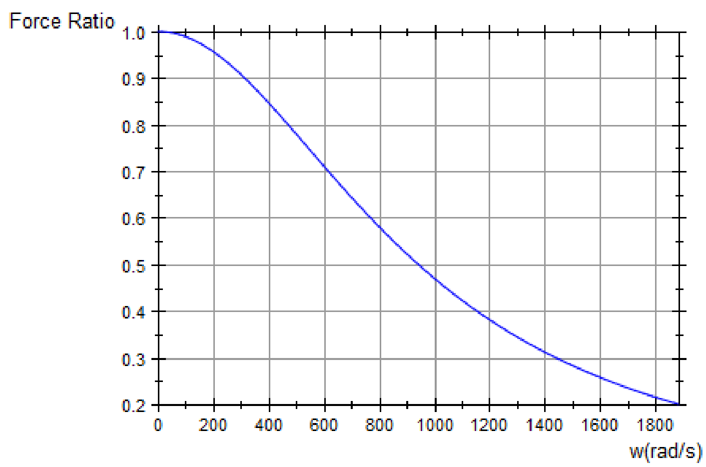
$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{F(t)}{k}$$

We know that $\zeta = 1$ (critical damping) and $\omega_n = 2\pi \times 150 = 300\pi$ rad/s so

3.2 Write down the formula for the vibration amplitude X_0, and hence find a formula for F_0^*/F_0 in terms of ω. Plot a graph of F_0^*/F_0 against ω, for $0 < \omega < 600\pi$.

$$X_0 = \frac{F_0 / k}{\left\{ \left(1 - \omega^2 / (300\pi)^2\right)^2 + \left(2\omega / 300\pi\right)^2 \right\}^{1/2}}$$

$$\Rightarrow \frac{kX_0}{F_0} = \frac{1}{\left\{ \left(1 - \omega^2 / (300\pi)^2\right)^2 + \left(2\varsigma\omega / 300\pi\right)^2 \right\}^{1/2}}$$



[4 POINTS]

3.2 Hence, determine the frequency range for which $\left| F_0^* / F_0 - 1 \right| < 0.05$ (i.e. 5% error in the force reading)

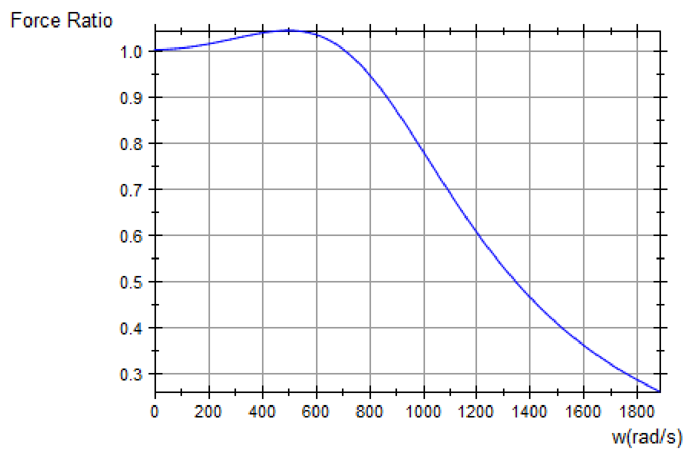
The lower limit of the frequency range is zero, the upper limit is the frequency for which

$$1 - \frac{1}{\left\{ \left(1 - \omega^2 / (300\pi)^2\right)^2 + \left(2\omega / 300\pi\right)^2 \right\}^{1/2}} = 0.05$$

This equation can easily be solved (by hand if you are a glutton for punishment, or mupad).
The solution is $\omega = 300\pi / \sqrt{19} = 216$ rad/s=34Hz.

3.3 This frequency range can be improved by reducing the damping (to see this, plot the graph in 3.1 for ζ a bit less than 1. You don't have to submit this graph). Find the damping coefficient ζ that will maximize the frequency range for which $\left| F_0^* / F_0 - 1 \right| < 0.05$ and determine the corresponding frequency range.

Here's a plot with lower damping



Now the measurement slightly over-estimates the force, before dropping off. We can maximize the frequency range by allowing the maximum value of the force ratio to just hit 5% error. We need to first solve for the frequency at which the max occurs (differentiate, set to zero and solve – use mupad)

The max occurs at $\omega = \omega_n \sqrt{1-2z^2}$
Substituting back into the force ratio we see that the max value is

$$\left[\frac{F_0^*}{F_0}\right]_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Now set the max value equal to 1.05 and solve for ζ , which gives $\zeta = 0.589$.

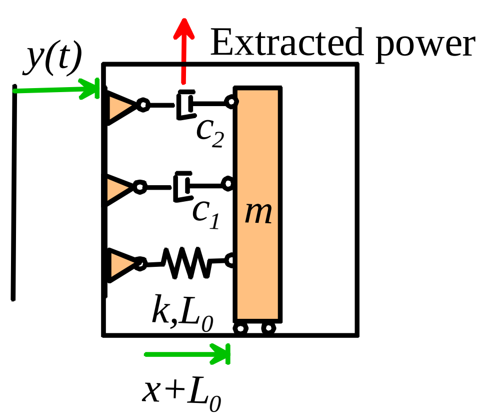
Now to find the frequency range, use this value of ζ to calculate the frequency where the signal drops to 5% below 1:

$$1 - \frac{1}{\left\{\left(1-\omega^2/(300\pi)^2\right)^2 + \left(2\times 0.589\times\omega/300\pi\right)^2\right\}^{1/2}} = 0.05$$

This can be solved (mupad) to see that $\omega = 819$ rad/s.

[4 POINTS]

4. The figure shows a simplified idealization of a micro-scale energy harvesting device intended to scavenge energy from the motion of an insect (from Aktakka *et al* 2011). The damper c_1 represents 'parasitic damping' (e.g. from friction or air resistance); the damper c_2 represents (e.g.) the effects of a magnet moving through a coil, generating an electrical current that can do useful work.

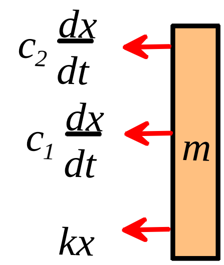


The goal of this problem is to derive the equations that are used to design an optimized energy harvesting device.

4.1 Use Newton's law of motion to show that the equation of motion for the length of the spring/dampers x can be arranged into the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2(\zeta_1+\zeta_2)}{\omega_n}\frac{dx}{dt}+x=-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

Give formulas for $\omega_n, \zeta_1, \zeta_2, K$.



A FBD is shown. Note that the acceleration of the mass is $\dfrac{d^2(x+y)}{dt^2}$, so that

$$m\frac{d^2(x+y)}{dt^2}=-(c_1+c_2)\frac{dx}{dt}-kx$$

$$\Rightarrow \frac{m}{k}\frac{d^2x}{dt^2}+\frac{(c_1+c_2)}{k}\frac{dx}{dt}+x=-\frac{m}{k}\frac{d^2y}{dt^2}$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2(\zeta_1+\zeta_2)}{\omega_n}\frac{dx}{dt}+x=-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

$$\omega_n=\sqrt{\frac{k}{m}} \qquad \zeta_1=\frac{c_1}{2\sqrt{km}} \qquad \zeta_2=\frac{c_2}{2\sqrt{km}} \qquad K=1$$

[2 POINTS]

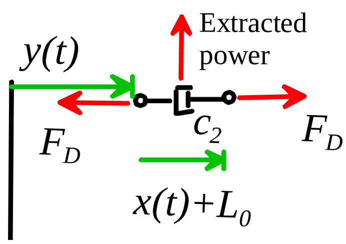
4.2 Assume that the device vibrates harmonically, so that $y(t)=Y_0\sin\omega t$. Write down the formula for the vibration amplitude X_0 in terms of $\omega_n, \zeta_1, \zeta_2, K$ and Y_0

This is solution 6 from the handout – the amplitude is

$$X_0=\frac{KY_0\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2(\zeta_1+\zeta_2)\omega/\omega_n\right)^2\right\}^{1/2}}$$

4.3 The power extracted from the electromagnetic coil is equal to the rate of work done by the forces acting on the damper c_2 . Show that the power can be expressed as

$$P(t)=F_D\frac{dx}{dt}=c_2\left(\frac{dx}{dt}\right)^2$$



The force on the damper is

$$F_D=c_2\left(\frac{dx}{dt}\right)$$

The total work done by the forces acting on the damper is

$$F_D\frac{d(y+x)}{dt}-F_D\frac{dy}{dt}=F_D\frac{dx}{dt}=c_2\left(\frac{dx}{dt}\right)^2$$

[2 POINTS]

4.4 Hence, show that instantaneous power generated is $P(t) = c_2 X_0^2 \omega^2 \cos^2(\omega t + \phi)$

We know that $x = X_0 \sin(\omega t + \phi)$. Therefore

$$\frac{dx}{dt} = \omega \cos(\omega t + \phi)$$

$$\Rightarrow P(t) = c_2 \left(\frac{dx}{dt} \right)^2 = c_2 X_0^2 \omega^2 \cos^2(\omega t + \phi)$$

[1 POINT]

4.5 The average power generated by the device can be computed by averaging the instantaneous power over one period of vibration, i.e.

$$\bar{P} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} P(t) dt$$

Use this expression and the results of 4.4, 4.2 to show that

$$\bar{P} = m Y_0^2 \omega^3 \frac{\zeta_2 \left(\dfrac{\omega}{\omega_n} \right)^3}{\left(1 - \dfrac{\omega^2}{\omega_n^2} \right)^2 + \left(2(\zeta_1 + \zeta_2) \dfrac{\omega}{\omega_n} \right)^2}$$

Evaluating the integral gives

$$\bar{P} = c_2 X_0^2 \omega^2 \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos^2(\omega t + \phi) dt$$

$$= \frac{1}{2} c_2 X_0^2 \omega^2$$

$$= \frac{1}{2} c_2 \omega^2 \frac{Y_0^2 \left(\omega^2 / \omega_n^2 \right)^2}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + \left(2(\zeta_1 + \zeta_2) \omega / \omega_n \right)^2 \right\}}$$

$$= \frac{1}{2} \zeta_2 2\sqrt{km} \omega^2 \frac{\omega}{\omega_n} \frac{Y_0^2 \left(\omega / \omega_n \right)^3}{\left\{ \left(1 - \omega^2 / \omega_n^2 \right)^2 + \left(2(\zeta_1 + \zeta_2) \omega / \omega_n \right)^2 \right\}}$$

$$= m Y_0^2 \omega^3 \frac{\zeta_2 \left(\dfrac{\omega}{\omega_n} \right)^3}{\left(1 - \dfrac{\omega^2}{\omega_n^2} \right)^2 + \left(2(\zeta_1 + \zeta_2) \dfrac{\omega}{\omega_n} \right)^2}$$

[4 POINTS]

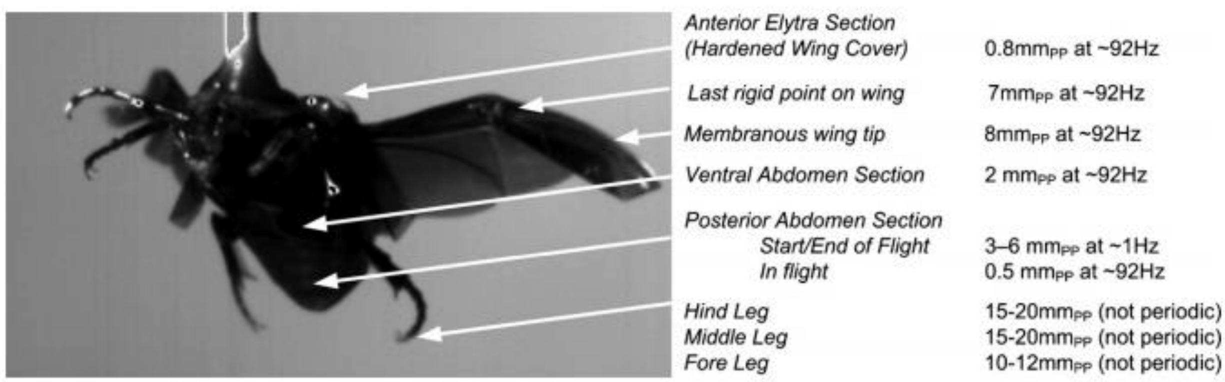
4.6 For a fixed excitation frequency ω (e.g. the frequency of the insect's wing beat), the power is maximized (for small damping) by tuning the natural frequency of the spring-mass system to the same frequency ($\omega = \omega_n$) . With this choice, show that the power is maximized when $\zeta_2 = \zeta_1$, and determine an expression for the optimal power, in terms of m, Y_0, ζ_1, ω .

For $\omega = \omega_n$ the power is

$$\bar{P} = mY_0^2\omega^3 \frac{\zeta_2}{\left(2(\zeta_1 + \zeta_2)\right)^2}$$

Now to maximize we can differentiate with respect to ζ_2 , set the derivative to zero and solve – this gives $\zeta_1 = \zeta_2$. The power is

$$\bar{P} = \frac{1}{16\zeta_1} mY_0^2\omega^3$$



4.7 The figure shows measured frequencies and amplitudes of vibration from a Green June Beetle. Estimate the power that could be harvested from the bug assuming a vibration amplitude of 1mm, frequency of 92Hz, mass of 0.13gram (10% of the beetle's mass), and damping $\zeta = 0.1$

Substituting numbers gives 15 milliWatts (i.e. you would need about 1000 bugs to power a light-bulb…)

[1 POINT]